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NOTE

# Weighted Norm Inequalities for Pluriharmonic Conjugate Functions

Jaesung Lee

*Department of Mathematics, Sogang University, Seoul 121-742, Korea*
E-mail: jalee@sogang.ac.kr

and

Kyung Soo Rim

*School of Mathematics, Korea Institute for Advanced Study, Seoul 130-012, Korea*
E-mail: ksrim@math.kias.re.kr

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We define pluriharmonic conjugate functions on the unit ball of $\mathbb{C}^n$. Then we show that for a weight there exist weighted norm inequalities for pluriharmonic conjugate functions on $L^p$ if and only if the weight satisfies the $A_p$-condition. As an application, we prove the equivalence of the weighted norm inequalities for the Cauchy integral and the $A_p$-condition of the weight. Along the way, we show that there exist norm inequalities for pluriharmonic conjugate functions on $BMO$ and on the nonisotropic Lipschitz spaces. 

*Key Words:* weighted norm inequality; pluriharmonic conjugate function.

## 1. INTRODUCTION

Let $B$ be the unit ball of $\mathbb{C}^n$, let $S$ be the unit sphere of $\mathbb{C}^n$, and let $\sigma$ be the rotation invariant, positive Borel measure on $S$ with $\sigma(S) = 1$. For $1 \leq p \leq \infty$, we denote $L^p = L^p(S, \sigma)$ the usual Lebesgue space on $S$, $H^p = H^p(B) = H^p(S)$ the Hardy space, and $BMOA = H^2 \cap BMO$ where $BMO$ is the collection of bounded mean oscillations on $S$. For $f \in L^1$, we



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define two integral operators $Jf$ and $Kf$ on $B$ as

$$(1.1)\qquad \begin{aligned} Jf(z) &= \textstyle\int_S J(z,\xi)f(\xi)\,d\sigma(\xi) \\ \text{and}\qquad Kf(z) &= \textstyle\int_S K(z,\xi)f(\xi)\,d\sigma(\xi), \end{aligned}$$

where $J(z,\xi) = 2\mathrm{Re}\,C(z,\xi)$ and $K(z,\xi) = 2\mathrm{Im}\,C(z,\xi)$. Here $C(z,\xi) = (1-\langle z,\xi\rangle)^{-n}$ is the Cauchy kernel for $B$ [8, Definition 3.2.1]. If $f \in H^1$ with $f(0)=0$, then $J\mathrm{Re}\,f = \mathrm{Re}\,f$ and $K\,\mathrm{Re}\,f = \mathrm{Im}\,f$, which can be easily obtained by the reproducing property of the Cauchy integrals of $f$ and $\bar{f}$ [8, Theorem 3.2.5]. Each kernel $J(z,\xi)$, $K(z,\xi)$ is pluriharmonic in $z$ if its variable $\xi$ is fixed. For this reason, we call $Jf$ the pluriharmonic integral of $f$ and $Kf$ the pluriharmonic conjugate integral of $f$, respectively. Throughout the paper, we denote the radial limit of a function $h$ on $B$ by $h^*$. In particular, the symbol $\tilde{f}$ denotes $(Kf)^*$ and we call it the pluriharmonic conjugate function of $f$. Since the radial limit of the Cauchy integral $Cf$ exists almost everywhere [8, Theorem 6.2.3], the pluriharmonic conjugate function is well defined for $f \in L^1$.

Now considered a classical result of conjugate functions, in 1924, M. Riesz proved that for $n=1$ and $1<p<\infty$, harmonic conjugate functions are bounded on $L^p$ [2, Theorem 2.3 of Chap. 3]. The past eight decades have witnessed many results of research on this subject. Here we mention only some of them. Two books [2, 3] also provide good references on that subject. In 1973, Hunt *et al.* [4] proved that harmonic conjugate functions are bounded on weighted measured Lebesgue spaces. For the Bergman spaces, Dzhrbashyan and Karapetyan [1] obtained the norm inequalities for pluriharmonic conjugate functions. More recently, Rim and Choi [6] got the weighted norm inequalities for $\mathcal{M}$-harmonic conjugate functions for $p=2$.

In the present paper, we prove the following:

1. Given a weight function on $S$ and $1<p<\infty$, there exist weighted norm inequalities for pluriharmonic conjugate functions on $L^p$ if and only if the weight satisfies the $A_p$-condition.

2. Given a weight function on $S$ and $1<p<\infty$, there exist weighted norm inequalities for the Cauchy integral on $L^p$ if and only if the weight satisfies the $A_p$-condition.

3. There exist norm inequalities for pluriharmonic conjugate functions on the nonisotropic Lipschitz space and on $BMO$.

As corollaries, in $H^p$ for $1<p<\infty$, the real part and the imaginary part of a function which fixes the origin are weighted norm equivalent. And in $BMOA$, the real part and the imaginary part of a function which fixes the origin are norm equivalent.

## 2. $A_p$-WEIGHTS

Let $\omega$ be a nonnegative locally integrable function on $S$. For $p > 1$, we say that $\omega$ satisfies the $A_p$-condition (or $\omega$ is an $A_p$-weight) if

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q \omega \, d\sigma \left( \frac{1}{\sigma(Q)} \int_Q \omega^{-1/(p-1)} \, d\sigma \right)^{p-1} < \infty, \tag{2.1}$$

where $Q = Q(\xi, \delta) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle|^{1/2} < \delta\}$ is a non-isotropic ball of $S$. The form (2.1) is related to the oscillation of functions. In complex one dimension, one can easily obtain that $\log \omega \in BMO$ if and only if $\omega^\alpha$ is an $A_2$-weight for some $\alpha > 0$ [2, p. 258].

2.1. LEMMA. *For $1 < p < \infty$ and $\alpha > 1$, there are constants $c$ and $c'$ (depending on $p$ and $\alpha$) such that for any $f \in L^p$*

$$\|M_\alpha Jf\|_p \leq c\|f\|_p \qquad \textit{and} \qquad \|M_\alpha Kf\|_p \leq c'\|f\|_p,$$

*where $M_\alpha h$ is the maximal function of $h$ on the Koranyi's admissible region $D_\alpha$.*

*Proof.* We may assume that $f$ is real valued. Observe that $M_\alpha Jf$ and $M_\alpha Kf$ are less than or equal to $2M_\alpha Cf$. The claim holds from the fact that the maximal function of the Cauchy integral is bounded in $L^p$ [8, Theorem 6.3.1]. ∎

2.2. COROLLARY. *For $1 < p < \infty$, there are positive constants $c_p$ and $c'_p$ such that*

$$c_p\|\mathrm{Re}\, f\|_p \leq \|\mathrm{Im}\, f\|_p \leq c'_p\|\mathrm{Re}\, f\|_p,$$

*for all $f \in H^p$ with $f(0) = 0$.*

*Proof.* Since $K\mathrm{Re}\, f = \mathrm{Im}\, f$, by Lemma 2.1 the corollary holds. ∎

Now we will prove the necessity of the first result.

2.3. THEOREM. *Let $\omega$ be nonnegative integrable function on $S$. For $1 < p < \infty$, if there is a constant $c_p$ such that for $f \in L^p$*

$$\int_S |\tilde{f}|^p \omega \, d\sigma \leq c_p \int_S |f|^p \omega \, d\sigma, \tag{2.2}$$

*then $\omega$ satisfies the $A_p$-condition.*

*Proof.* By Lemma 2.1, we can replace $f$ by $\tilde{f}$ in inequality (2.2). A property of the conjugate functions yields that

$$\int_S |(Jf)^*|^p \omega\, d\sigma = \int_S |\tilde{\tilde{f}}|^p \omega\, d\sigma \\ \le c_p^2 \int_S |f|^p \omega\, d\sigma.$$

So the assumption implies the weighted norm inequalities for $Jf$. For each $\xi$, we define the modification functions $R_1(\eta, \xi)$ and $R_2(\eta, \xi)$ on $S$ by

$$R_1(\eta, \xi) = \begin{cases} 1 & \text{if } \mathrm{Re}\,(1 - \langle \eta, \xi \rangle)^n \ge 0, \\ -1 & \text{otherwise} \end{cases}$$

and

$$R_2(\eta, \xi) = \begin{cases} 1 & \text{if } \mathrm{Im}\ (1 - \langle \eta, \xi \rangle)^n \ge 0, \\ -1 & \text{otherwise.} \end{cases}$$

Suppose $Q_1$ and $Q_2$ are nonisotropic balls with radius sufficiently small $\delta$ and that they are contained in another small nonisotropic ball with radius $3\delta$. Choose a nonnegative $f$ supported in $Q_1$. Then for almost all $\xi \in Q_2$ we have

$$(JfR_1)^*(\xi) + \widetilde{fR_2}(\xi) = \int_{Q_1} \frac{2\big(|\mathrm{Re}(1 - \langle \eta, \xi \rangle)^n| + |\mathrm{Im}\,(1 - \langle \eta, \xi \rangle)^n|\big)}{|1 - \langle \xi, \eta \rangle|^{2n}} \\ \times f(\eta)\, d\sigma(\eta) \\ \ge \int_{Q_1} \frac{2}{|1 - \langle \xi, \eta \rangle|^n} f(\eta)\, d\sigma(\eta) \\ \ge 0.$$

By the selection of nonisotropic balls, $\sigma(Q_1) \approx \delta^{2n}$, so there is an absolute constant $c$ such that for almost all $\xi \in Q_2$

$$\big((JfR_1)^*(\xi) + \widetilde{fR_2}(\xi)\big)^p \ge c\left(\frac{1}{\sigma(Q_1)} \int_{Q_1} f\, d\sigma\right)^p. \tag{2.3}$$

Throughout the proof, we use the letter $c$ which denotes a positive constant whose value may change from line to line. It depends on $n$ and possibly on $p$ but does not depend on $\delta$. Setting $f = \chi_{Q_1}$ and integrating (2.3) over $Q_2$ after being multiplied by $\omega$, we get

$$\int_{Q_2} \omega\, d\sigma \le c \int_{Q_2} \big(|J(fR_1)^*|^p + |\widetilde{fR_2}|^p\big) \omega\, d\sigma \\ \le c \int_S \big(|f(\xi) R_1(\xi, \xi)|^p + |f(\xi) R_2(\xi, \xi)|^p\big) \omega(\xi)\, d\sigma(\xi) \\ \le 2c \int_{Q_1} \omega\, d\sigma. \tag{2.4}$$

Next putting $f = \chi_{Q_2}$ and integrating (2.3) over $Q_1$ after being multiplied by $\omega$, we have

$$\int_{Q_1} \omega \, d\sigma \leq 2c \int_{Q_2} \omega \, d\sigma. \tag{2.5}$$

From (2.4) and (2.5), the integrals of $\omega$ over $Q_1$ and $Q_2$ are equivalent.

Now put $f = \omega^{\alpha} \chi_{Q_1}$ in (2.3) (the constant $\alpha$ will be chosen later) and integrate (2.3) over $Q_2$. Then

$$\left( \frac{1}{\sigma(Q_1)} \int_{Q_1} \omega^{\alpha} \, d\sigma \right)^p \int_{Q_2} \omega \, d\sigma \leq c \int_{Q_1} \omega^{\alpha p + 1} \, d\sigma. \tag{2.6}$$

Finally take $\alpha = -1/(p-1)$ and apply (2.4) and (2.5) to (2.6); then there is a constant $c$ such that

$$\frac{1}{\sigma(Q_1)} \int_{Q_1} \omega \, d\sigma \left( \frac{1}{\sigma(Q_1)} \int_{Q_1} \omega^{-1/(p-1)} \, d\sigma \right)^{p-1} \leq c < \infty,$$

where the constant $c$ depends only on $p$. Consequently, we have the desired $A_p$-condition. ∎

## 3. THE SHARP MAXIMAL FUNCTIONS

In this section we will prove the converse of Theorem 2.3. To start with, we introduce a lemma. For the proof of it, see [8, Lemma 6.1.1].

3.1. LEMMA. *Let $\xi, \eta, \omega \in S$. Suppose $d(\omega, \eta) < \delta$ and $d(\omega, \xi) > 2\delta$. Then we have*

$$|C(\xi, \eta) - C(\xi, \omega)| \leq \frac{c\delta}{|1 - \langle \xi, \omega \rangle|^{n+1/2}},$$

*where $c$ is an absolute constant.*

Let $p > 0$. For a locally integrable function $f$ on $S$, the sharp maximal function $f^{\#^p}$ is defined for $\xi \in S$ by setting

$$f^{\#^p}(\xi) = \sup_{\xi \in Q} \left( \frac{1}{\sigma(Q)} \int_Q |f - f_Q|^p \, d\sigma \right)^{1/p},$$

where the supremum is taken over all the nonisotropic balls $Q$ containing $\xi$ and $f_Q$ stands for the average of $f$ over $Q$. The sharp maximal operator $f \mapsto f^{\#^p}$ is an analogue of the Hardy–Littlewood maximal operator $M$. Note that $f^{\#^p}(\xi) \leq 2Mf(\xi)$. The next result plays an important role in the computation of the weighted norm inequalities.

3.2. THEOREM. *Let $f \in L^1$. For $q > 1$, there is a constant $c_q$ such that*

$$\tilde{f}^{\#^1}(\xi) \le c_q f^{\#^q}(\xi),$$

*for almost all $\xi$.*

*Proof.* Fix $Q = Q(\xi_Q, \delta)$. It suffices to show that for each $q > 1$ there are constants $\lambda = \lambda(Q, f)$ and $c_q$ depending only on $q$ such that

$$\frac{1}{\sigma(Q)} \int_Q |\tilde{f}(\xi_Q) - \lambda| \, d\sigma \le c_q f^{\#^q}(\xi_Q). \tag{3.1}$$

Put

$$\begin{aligned} f &= (f - f_Q)\chi_{2Q} + (f - f_Q)\chi_{S\setminus 2Q} + f_Q \\ &= f_1 + f_2 + f_Q, \qquad \text{we define.} \end{aligned}$$

Since $\widetilde{f_Q} = 0$, we have

$$\tilde{f} = \tilde{f}_1 + \tilde{f}_2.$$

Put $g = Jf_2 + iKf_2$. Then it is continuous on $B \cup Q$. Now letting $\lambda = ig(\xi_Q)$ in (3.1), we shall prove the claim. The integral in (3.1) is estimated as

$$\begin{aligned} \int_Q |\tilde{f} + ig(\xi_Q)| \, d\sigma &\le \int_Q |\tilde{f}_1| \, d\sigma + \int_Q |\tilde{f}_2 + ig(\xi_Q)| \, d\sigma \\ &= I_1 + I_2, \quad \text{we define.} \end{aligned}$$

We first estimate $I_1$ : By Hölder's inequality we get

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q |\tilde{f}_1| \, d\sigma &\le \left( \frac{1}{\sigma(Q)} \int_Q |\tilde{f}_1|^q \, d\sigma \right)^{1/q} \\ &\le \frac{c_q}{\sigma(Q)^{1/q}} \|f_1\|_q, \end{aligned}$$

where the last inequality follows from Lemma 2.1. Now we have

$$\begin{aligned} \|f_1\|_q &= \left( \int_{2Q} |f - f_Q|^q \, d\sigma \right)^{1/q} \\ &\le \left( \int_{2Q} |f - f_{2Q}|^q \, d\sigma \right)^{1/q} + \sigma(2Q)^{1/q} |f_{2Q} - f_Q|. \end{aligned}$$

Thus by applying Hölder's inequality in the last term of the above, we see that there is a constant $c'_q$ such that

$$\frac{1}{\sigma(Q)} \int_Q |\tilde{f}_1| \, d\sigma \le c'_q f^{\#^q}(\xi_Q).$$

Now we estimate $I_2$ : Since $f_2 \equiv 0$ on $2Q$, we have

$$I_2 = \int_Q |f_2 + i\tilde{f}_2 - g(\xi_Q)|\, d\sigma$$
$$\leq \int_{S\backslash 2Q} |f_2(\eta)| \int_Q |(J+iK)(\xi,\eta) - (J+iK)(\xi_Q,\eta)|\, d\sigma(\xi)\, d\sigma(\eta),$$

where $(J + iK)(\xi, \eta) = J(\xi, \eta) + iK(\xi, \eta)$ for notational simplicity. By Lemma 3.1 we get an upper bound such that

$$\frac{1}{\sigma(Q)} I_2 \leq c\delta \int_{S\backslash 2Q} \frac{|f_2(\eta)|}{|1 - \langle \eta, \xi_Q\rangle|^{n+1/2}}\, d\sigma(\eta), \tag{3.2}$$

where $c$ is an absolute constant. Write $S\backslash 2Q = \bigcup_{k=1}^{\infty} 2^{k+1}Q \setminus 2^k Q$. Then the integral of (3.2) is equal to

$$\sum_{k=1}^{\infty} \int_{2^{k+1}Q\backslash 2^k Q} \frac{|f(\eta) - f_Q|}{|1 - \langle \eta, \xi_Q\rangle|^{n+1/2}}\, d\sigma(\eta)$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{(2n+1)k}\delta^{2n+1}} \left( \int_{2^{k+1}Q} |f - f_{2^{k+1}Q}|\, d\sigma \right.$$
$$\left. + \sum_{j=0}^{k} \int_{2^{k+1}Q} |f_{2^{j+1}Q} - f_{2^j Q}|\, d\sigma \right).$$

Thus there is a constant $c$ such that

$$\frac{1}{\sigma(Q)} \int_Q |\tilde{f}_2 + ig(\xi_Q)|\, d\sigma \leq c \sum_{k=1}^{\infty} \frac{k}{2^k} Mf^{\#^1}(\xi_Q),$$

as desired. ∎

By observing Theorem 3.2, we know that $\tilde{f}$ is bounded on $BMO$ and there are positive constants $c$ and $c'$ such that

$$c\|\mathrm{Re}\, f\|_{BMO} \leq \|\mathrm{Im}\, f\|_{BMO} \leq c'\|\mathrm{Re}\, f\|_{BMO},$$

for all $f \in BMOA$ with $f(0) = 0$. The proof of the follwing lemma is essentially the same as that of Theorem 2.20 of [3]. (The only difference between these two is the domain of functions and the following lemma is independent of the theory of conjugate function.) So we omit its proof.

3.3. LEMMA. *For $0 < p < \infty$, let $\omega$ be an $A_p$-weight. Then there is a constant $c_p$ such that*

$$\int_S (Mf)^p \omega\, d\sigma \leq c_p \int_S (f^{\#^1})^p \omega\, d\sigma,$$

*for all $f \in L^p$.*

The next theorem is the converse of Theorem 2.3.

3.4. THEOREM. *For* $1 < p < \infty$, *if* $\omega$ *is an* $A_p$*-weight, then there is a constant* $C_p$ *such that*

$$\int_S |\tilde{f}|^p \omega \, d\sigma \le C_p \int_S |f|^p \omega \, d\sigma,$$

*for all* $f \in L^p$.

*Proof.* Fix $p > 1$ and let $f \in L^p$. Then by Lemma 3.3 there is a constant $c_p$ such that

$$\int_S |\tilde{f}|^p \omega \, d\sigma \le \int_S |M\tilde{f}|^p \omega \, d\sigma \le c_p \int_S |\tilde{f}^{\#^1}|^p \omega \, d\sigma.$$

Take $q > 0$ such that $p/q > 1$. By Theorem 3.2, the last term of the above inequalities is bounded by some constant (depending on $p$ and $q$) times

$$\int_S |f^{\#^q}|^p \omega \, d\sigma \le c \int_S (M|f|^q)^{p/q} \omega \, d\sigma \le c' \int_S |f|^p \omega \, d\sigma,$$

where two constants $c$ and $c'$ depend on $p$ and $q$ and the second inequality follows from Lemma 3.3. So we finish the proof. ∎

3.5. COROLLARY. *For* $1 < p < \infty$, *let* $\omega$ *be an* $A_p$*-weight. Then there are positive constants* $c_p$ *and* $c'_p$ *such that*

$$c_p \int_S |\mathrm{Re}\, f|^p \omega \, d\sigma \le \int_S |\mathrm{Im}\, f|^p \omega \, d\sigma \le c'_p \int_S |\mathrm{Re}\, f|^p \omega \, d\sigma,$$

*for all* $f \in H^p$ *with* $f(0) = 0$.

*Proof.* Recall that $K\mathrm{Re}\, f = \mathrm{Im}\, f$. Applying Theorem 3.4, we prove the corollary. ∎

As an application of Theorems 2.3 and 3.4 we have the weighted norm inequalities for the Cauchy integral:

3.6. THEOREM. *Let* $1 < p < \infty$. *There exist weighted norm inequalities for the Cauchy integral on* $L^p$ *if and only if the weight satisfies the* $A_p$*-condition.*

*Proof.* Suppose $\omega$ is an $A_p$-weight. Note that (2.2) implies the weighted norm inequalities for $(Jf)$ by replacing $f$ by $\tilde{f}$ in (2.2). First assume that $f$ is real valued in $L^p$. Then by Theorem 3.4 there is a constant $c_p$ such that

$$\int_S |(2Cf)^*|^p \omega \, d\sigma \le \int_S |(Jf)^*|^p \omega \, d\sigma + \int_S |\tilde{f}|^p \omega \, d\sigma \le c_p \int_S |f|^p \omega \, d\sigma, \tag{3.3}$$

where $Cf$ is the Cauchy integral of $f$. When $f$ is complex valued, write $f = f_1 + if_2$ for real valued $f_1$ and $f_2$. Then also by (3.3)

$$\int_S |(2Cf)^*|^p \omega\, d\sigma \le \int_S |(2Cf_1)^*|^p \omega\, d\sigma + \int_S |(2Cf_2)^*|^p \omega\, d\sigma$$
$$\le 2c_p \int_S |f|^p \omega\, d\sigma.$$

Conversely suppose there exist weighted norm inequalities for the Cauchy integral on $L^p$. Assume that $f$ is real valued. By Theorem 2.3 it suffices to prove that the weighted norm inequalities for $Cf$ yield that for $Jf$ and that for $Kf$. Then by taking the real (or imaginary) part of the hypothesis, we have the weighted norm inequalities for $Jf$ $(Kf)$. For a complex valued $f$, splitting $f$ into $f_1 + if_2$ for real valued $f_1$ and $f_2$, we would prove the remains. Thus the proof is complete. ∎

We define nonisotropic Lipschitz functions on $S$ and then show that pluriharmonic conjugate functions are bounded on the space of nonisotropic Lipschitz functions. For $0 < \alpha \le 2$, we say that $f$ is a nonisotropic Lipschitz function of order $\alpha$ on $S$ if there exits a finite constant $\|f\|_{\mathrm{lip}_\alpha}$ such that

$$\sup_{\xi,\,\eta \in S} \frac{|f(\xi) - f(\eta)|}{d(\xi, \eta)^\alpha} = \|f\|_{\mathrm{lip}_\alpha},$$

and then we write $f \in \mathrm{lip}_\alpha$. Note that with this seminorm, $\mathrm{lip}_\alpha$ becomes a Banach space provided we identify functions which differ by a constant almost everywhere.

From the lemma below, one can see that the norm of a nonisotropic Lipschitz function is also related to its oscillation over $S$. Moreover, we observe that the $BMO$-norm of smooth functions $f$ is equal to some constant times $\lim_{\alpha \searrow 0} \|f\|_{\mathrm{lip}_\alpha}$.

3.7. LEMMA. *Let $0 < \alpha \le 2$ and let $f \in L^1$. Then the following quantities are equivalent.*

(a) $\|f\|_{\mathrm{lip}_\alpha}$.

(b)
$$\sup_{Q,\, \xi \in Q} \frac{|f(\xi) - f_Q|}{\sigma(Q)^{\alpha/2n}}.$$

(c)
$$\sup_Q \left( \frac{1}{\sigma(Q)^{1+\alpha p/2n}} \int_Q |f - f_Q|^p \, d\sigma \right)^{1/p} \qquad (1 \le p < \infty).$$

(d)

$$\sup_Q \frac{1}{\sigma(Q)^{1+\alpha/2n}} \int_Q |f - f_Q| \, d\sigma.$$

*Proof.* Suppose $f \in \mathrm{lip}_\alpha$. Let $Q$ be a nonisotropic ball having center $\xi$ and radius $\delta$. Then since $\sigma(Q) \approx \delta^{2n}$, we have

$$\begin{aligned} |f(\xi) - f_Q| &\leq \frac{1}{\sigma(Q)} \int_Q |f(\xi) - f(\eta)| \, d\sigma(\eta) \\ &\leq c\|f\|_{\mathrm{lip}_\alpha} \sigma(Q)^{\alpha/2n}, \end{aligned}$$

where the constant $c$ depends on $\alpha$. So (b) is less than or equal to a constant times (a). Integrating $|f(\zeta) - f_Q|^p$ over $Q$, we get that (c) is less than or equal to a constant times (b), and by putting $p = 1$ also (d) is less than or equal to a constant times (c). It remains to prove that (a) is less than or equal to a constant times (d). To do this we use the method of Meyers [5].

Fix $\xi$, $\eta \in S$. Take $Q = Q(\xi, \delta)$ and $\delta = 2|1 - \langle \xi, \eta \rangle|^{1/2}$. Then we get

$$|f(\xi) - f(\eta)| \leq |f(\xi) - f_Q| + |f_Q - f(\eta)| = I + II, \qquad \text{we define.}$$

We will only estimate $I$, since the estimate of $II$ is identical. Inductively, choose a sequence of nonisotropic balls $\{Q_k\}$ such that $k = 1, 2, 3, \ldots$,

$$\begin{aligned} Q_k &\searrow \{\xi\} \qquad \text{as} \quad k \to \infty, \\ \sigma(Q_k) &= \frac{1}{2}\sigma(Q_{k-1}), \\ Q_0 &= Q. \end{aligned}$$

Then

$$I \leq |f(\xi) - f_{Q_k}| + \sum_{j=1}^{k} |f_{Q_j} - f_{Q_{j-1}}| = I_1 + I_2, \qquad \text{say.}$$

As $k \to \infty$, $I_1$ goes to 0 for almost all $\zeta$. So it suffices to estimate $I_2$. Observe that

$$\begin{aligned} I_2 &\leq \sum_{j=1}^{k} \frac{1}{\sigma(Q_j)} \int_{Q_j} |f - f_{Q_{j-1}}| \, d\sigma \\ &\leq 2 \sum_{j=1}^{k} \frac{1}{\sigma(Q_{j-1})} \int_{Q_{j-1}} |f - f_{Q_{j-1}}| \, d\sigma \\ &= 2c\sigma(Q)^{\alpha/2n} \sum_{j=1}^{k} \frac{1}{2^{j\alpha/2n}}. \end{aligned}$$

Since $\delta = 2|1 - \langle \xi, \eta \rangle|^{1/2}$, we have $I_2 \le cd(\xi, \eta)^\alpha$. Thus we have $|f(\xi) - f(\eta)| \le cd(\xi, \eta)^\alpha$ for almost all $\xi$, $\eta$. Since $f$ is a representation of some equivalent class in $L^1(S)$, we can redefine $f$ so that

$$|f(\xi) - f(\eta)| \le cd(\xi, \eta)^\alpha \qquad (\xi, \eta \in S).$$

Therefore the proof is complete. ∎

3.8. THEOREM. *There exist norm inequalities for pluriharmonic conjugate functions on* $\mathrm{lip}_\alpha$.

*Proof.* We can prove this theorem in the same way as the proof of Theorem 3.2. In (3.1), replace $\sigma(Q)$ by $\sigma(Q)^{1+\alpha/2n}$ and apply (c) of Lemma 3.7. Then we obtain the proof. ∎

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